Imperial College London

Lecture 8
Reactance and Frequency Response

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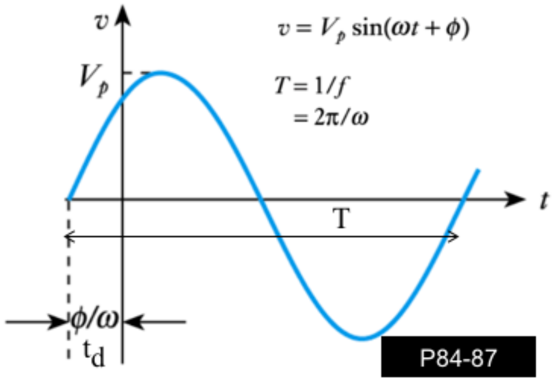
Sine and Cosine waves

- We have considered sine wave signals earlier in lectures and labs. We know a sinusoidal (or AC) voltage is given by the equation:

$$v(t) = V_p \sin(2\pi f t + \phi)$$

where V_p is the peak voltage
f is the frequency in Hz
Φ is the phase angle, either in radians or in degrees

- We often use ω, the angular frequency, instead of *f*, and ω = 2π *f*
- If Φ is in radians, then the time delay t_d is given by Φ/ω.
- Remember that period T = 1/f and one cycle of a sinewave corresponds to a phase angle of 2π radians or 360 degrees.



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Let us revisit sine and cosine waves. A sine wave can be completely defined with three parameters Vp, the peak voltage (or amplitude), its frequency w in radians/second or f in cycles/second (Hz), and the phase angle Φ.

Cosine waves are the same as sine waves, except that cosine wave is 90 degrees or π/2 radians advance in phase. That is:

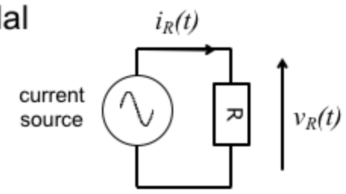
$$v(t) = V_p \sin(2\pi f t + \phi) \;\; = V_p \cos(2\pi f t + \phi + \frac{\pi}{2})$$

It is worth remembering that one cycle of a sine or cosine wave has a phase angle value of 2π radians or 360 degrees.

Sinewave through a resistor, capacitor and inductor

- Consider the resistor circuit driven by a sinusoidal current:

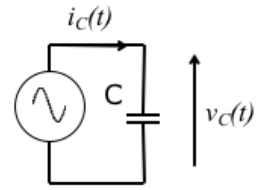
$$i_R(t) = I_P \sin \omega t$$



- Using Ohm's law, we have:

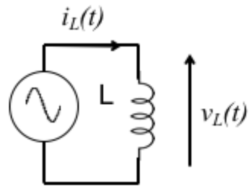
$$v_R(t) = i_R(t)R = I_P \times R \sin \omega t$$

- Now consider a capacitor driven by the same signal:



$$v_C(t) = \frac{1}{C}\int i_C(t)\,dt = \frac{1}{C}\int I_P \sin \omega t \, dt = -\frac{I_P}{\omega C}\cos \omega t$$

- Now consider an inductor driven by the same signal:



$$v_L(t) = L\frac{di_L(t)}{dt} = L\frac{d(I_P \sin \omega t)}{dt} = \omega L \times I_P \cos \omega t$$

P110-112

Resistors obey Ohm's Law. The ratio of voltage to current through a resistor remains a constant no matter what frequency is the signal.

This is not true with capacitors and inductors. As shown here the ratio V/I when driven by a sine wave current source results in a voltage that is a cosine wave.

Consider the case of a capacitor. The capacitor voltage $V_c(t)$ is INVERSELY proportional to the frequency of the sinusoidal signal. If $\omega \to \infty, V_c(t) \to 0$.

Conversely, for inductor, $\omega \to \infty, V_L(t) \to \infty$.

V/I relationship of capacitors and inductors

- Let us ignore phase angle for the moment, and compute peak voltage and peak currents in all three cases.
- Resistor:

$$\frac{v_R(t)_{\max}}{i_R(t)_{\max}} = \frac{(I_P R \sin\omega t)_{\max}}{(I_P \sin\omega t)_{\max}} = R$$

- Capacitor:

$$\frac{v_C(t)_{\max}}{i_C(t)_{\max}} = \frac{(-\frac{I_P}{\omega C}\cos\omega t)_{\max}}{(I_P \sin\omega t)_{\max}} = \frac{I_P/\omega C}{I_P} = \frac{1}{\omega C}$$

- Inductor:

$$\frac{v_L(t)_{\max}}{i_L(t)_{\max}} = \frac{(\omega L I_P \cos\omega t)_{\max}}{(I_P \sin\omega t)_{\max}} = \frac{\omega L I_P}{I_P} = \omega L$$

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Let us for the moment just consider ratio of the peak magnitudes of the voltage and the current in the three cases.

The ratio is simply R for resistor.

The ratio is $1/\omega C$ for capacitor.

The ratio is ωL for inductor.

Reactance of Capacitors and Inductors

- The ratio of voltage to current is a measure of how the component opposes the flow of electricity
- In a resistor, this ratio is the **resistance**
- In inductors and capacitors it is termed its **reactance**
- Reactance is given the symbol *X*. Therefore:

$$\text{Reactance of a capacitor, } X_C = \frac{1}{\omega C}$$

$$\text{Reactance of an inductor, } X_L = \omega L$$

- Units of reactance is ohms, same as resistance.
- It can be used in much the same way as resistance: $V = I X_C$ $V = I X_L$
- Example: A sinusoidal voltage of 5 V peak and 100 Hz is applied across an inductor of 25 mH. What will be the peak current?

$$X_L = \omega L = 2\pi f L = 2 \times \pi \times 100 \times 25 \times 10^{-3} = 15.7\Omega$$

$$\text{Therefore} \quad I_L = \frac{V_L}{X_L} = \frac{5}{15.7} = 0.318A \ \text{(peak)}$$

For capacitors and inductors, this ratio of peak voltage over peak current is frequency dependent. They are called reactance.

Both resistance and reactance are measures of how the components oppose the flow of current. The unit of reactance is the same as that of resistance – in ohms.

We use the symbol X to represent reactance here.

Impedances
Circuits with Resistors and Capacitors

- Reactance can be viewed as “resistance” for a capacitor or an inductor
- Reactance only consider peak voltage and current in a component
- Current and voltage through a capacitor is related by the equations (Slide 3)

$$i_C = I_P \sin \omega t \qquad v_C = -\frac{I_P}{\omega C} \cos \omega t$$

Impedance: $$\frac{v_C}{i_C} = -\frac{1}{\omega C}\frac{\cos \omega t}{\sin \omega t} = \frac{1}{j\omega C}$$

- The ratio of voltage to current in a capacitor is now a complex number
- This ratio is known as impedance of the capacitor
- The use of complex number allows use to treat capacitors (and inductors) in a similar way to resistor – all analysis we used for resistors also works here, as long as we use complex number in our calculation

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Reactance is the equivalent of “resistance” in a resistor, except two things: 1) reactance is for capacitor or inductor; 2) reactance magnitude changes with frequency of the electrical signal.

Furthermore, when we use reactance, we only consider peak voltage and current in a component because we only use the maximum (or peak) voltage and current in the calculation. However, for two given voltages (or currents), the times when the peak occurs do not usually align in time. Therefore, how can we use reactance in nodal analysis? The answer is to use complex number representation.

When you charge a capacitor with sinewave current, the voltage is a cosine wave at the same frequency (see slide 3). In other words, the current and the voltage through a capacitors are related: they differs from each other by 90 degrees or π/2 radians.

$$\sin \omega t = \cos(\omega t + \frac{\pi}{2})$$

Let us now introduce a modified reactance to take this into account: the idea of impedance.

Impedance of a capacitor: $\frac{v_C}{i_C} = \frac{1}{j\omega C}$

Impedance of an inductor: $\frac{v_L}{i_L} = j\omega L$

Gain of a Two-port Networks

- While the properties of a pure ***resistance*** are not affected by the frequency of the signal concerned, this is not true of ***reactive*** components.
- We will start with a few basic concepts and then look at the characteristics of simple combinations of resistors, capacitors and inductors.
- A two-port network has two ports: an input port, and an output port.
- We can define voltages and currents at the input and output as shown here.
- Then:

$$\text{voltage gain } (A_v) = \frac{V_O}{V_i}$$

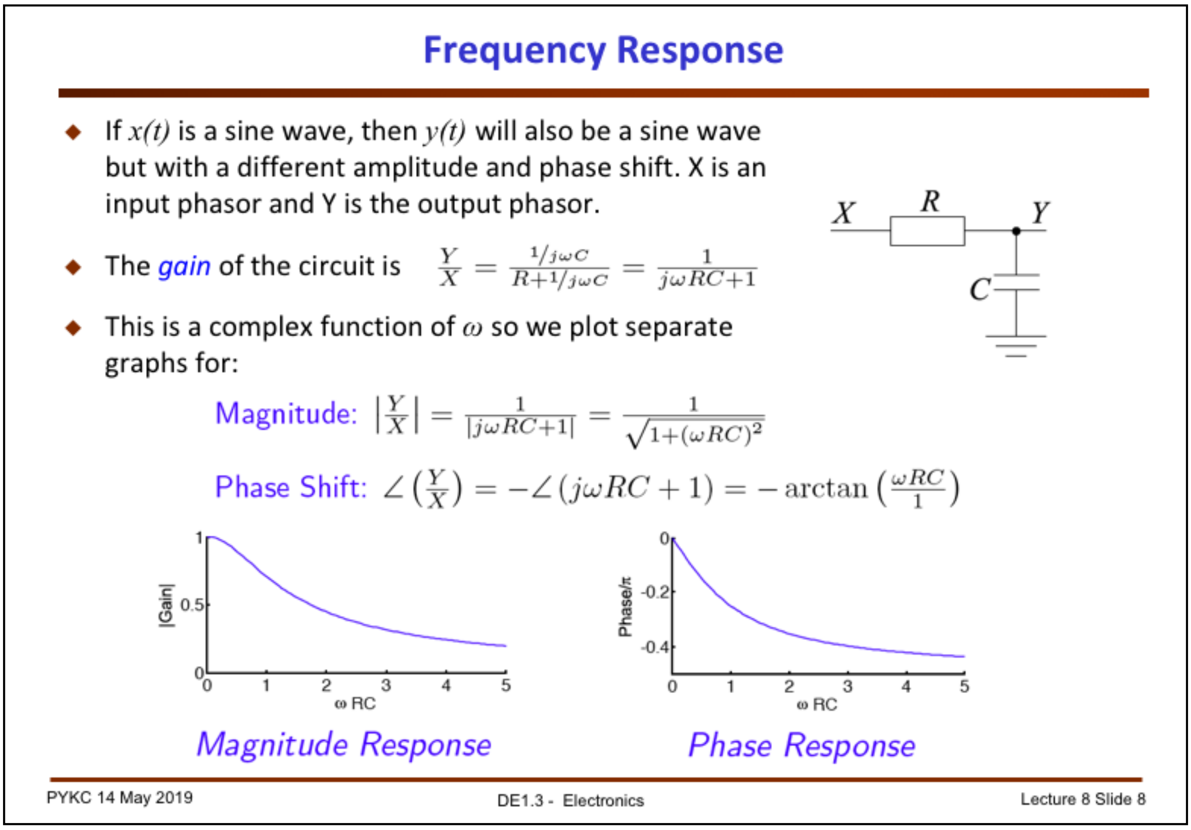
$$\text{current gain } (A_i) = \frac{I_O}{I_i}$$

$$\text{power gain } (A_p) = \frac{P_O}{P_i}$$

Reactive components are things like capacitors and inductors. Their reactance (equivalent to resistance in resistors) are frequency dependent. This frequency dependency turns out to be very useful in building interesting circuits, e.g. filters which provide frequency selectivity.

Before we look at these circuits, I want to introduce to idea of two port network and gain.

A two-port network as an input port to which we apply stimulus V_i. There is an output port that provides a signal V_O. The ratio V_O/V_I is the voltage gain.



Using complex number algebraic methods, we can easily work out the voltage gain of this simple RC network. Note that the gain equation is frequency dependent (i.e. it is a function of ω).

The relationship between output Y and input X as a function of signal frequency ω is known as **frequency response**.

From the gain equation, we can compute the magnitude of the gain as a function of frequency. We can also plot the phase difference (output relative to input) as a function of frequency. The former is known as the magnitude (or amplitude) response. The latter is known as the phase response.

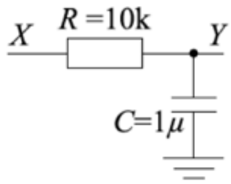
In the literature, graphs showing gain magnitude and phase vs frequency is also known as "**Bode diagrams**".

On this course, we will mainly focus on the magnitude response, and we will ignore the phase response most of the time.

Sine Wave Response

$RC = 10\,\text{ms}$

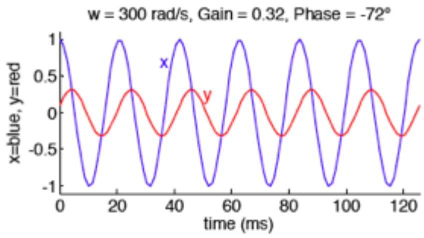
$\frac{Y}{X} = \frac{1}{j\omega RC+1} = \frac{1}{0.01j\omega+1}$

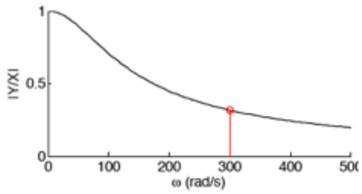


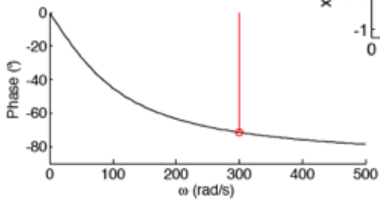
$\omega = 50 \Rightarrow \frac{Y}{X} = 0.89\angle -27°$

$\omega = 100 \Rightarrow \frac{Y}{X} = 0.71\angle -45°$

$\omega = 300 \Rightarrow \frac{Y}{X} = 0.32\angle -72°$





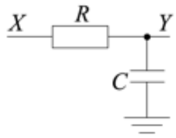


- The output, $y(t)$, lags the input, $x(t)$, by up to $90°$.

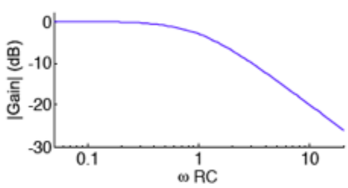
Here is a circuit with R = 10k and C = 1 microfarad, and how the gain magnitude and phase changes with frequency. Note that at $\omega = 1/RC$, the gain is 0.71 (or $1/\sqrt{2}$). This will come up again later.

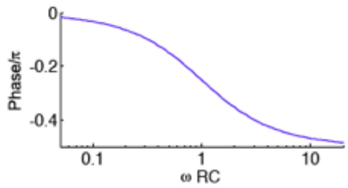
Logarithmic axes

- We usually use logarithmic axes for frequency and gain (but not phase) because % differences are more significant than absolute differences.
- E.g. 5 kHz versus 5.005 kHz is less significant than 10Hz versus 15Hz even though both differences equal 5Hz.
- Logarithmic voltage ratios are specified in *decibels* (dB) = $20 \log_{10} |V_2 / V_1|$.
- Common voltage ratios:



$\frac{\lvert V_2 \rvert}{\lvert V_1 \rvert}$	0.1	0.5	$\sqrt{0.5}$	1	$\sqrt{2}$	2	10	100
dB	−20	-6	-3	0	3	6	20	40





Note that 0 does not exist on a log axis and so the starting point of the axis is arbitrary.

Note: $P \propto V^2 \Rightarrow$ decibel power ratios are given by **10** $\log_{10} (P_2 / P_1)$

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Percentage differences are often more important than absolute differences. Therefore it is often more revealing if we plot frequency and gain in **logarithmic scales**.

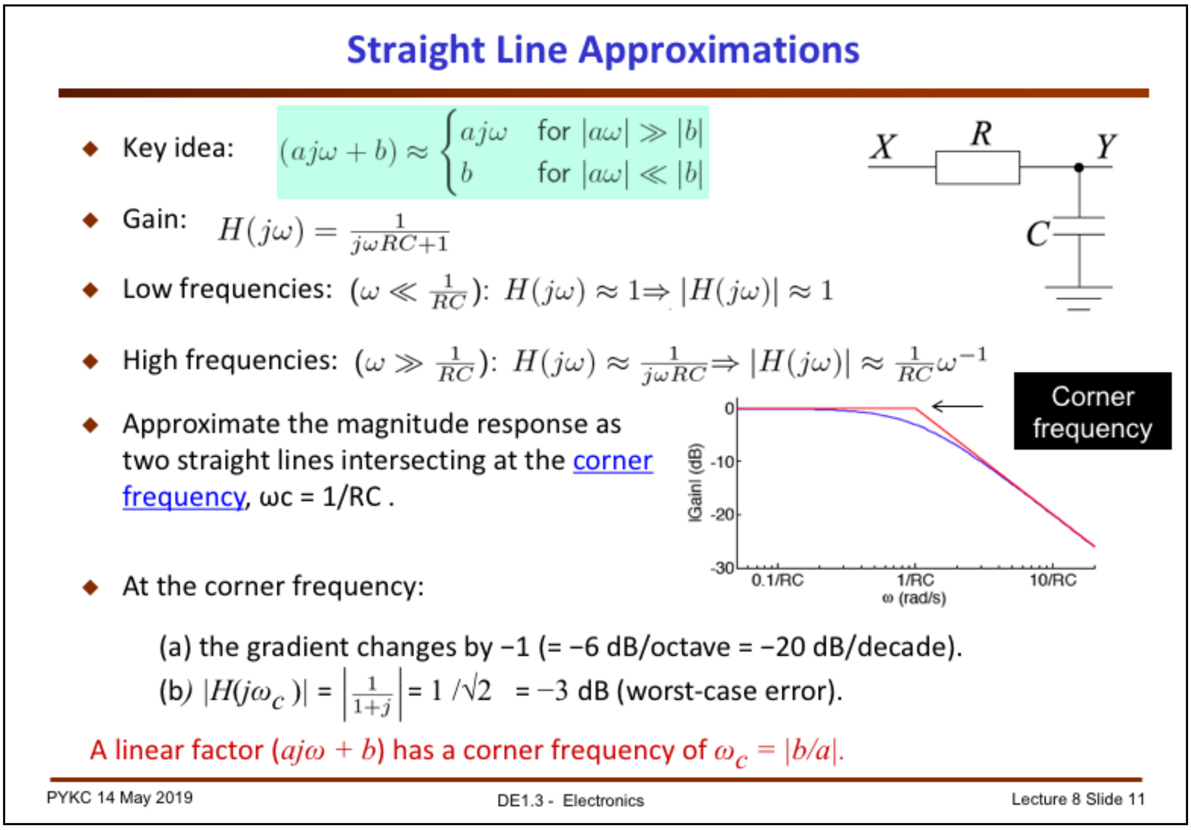
For voltages, we normally calculate the gain in terms of decibels (dB) as defined here.

Remember that dB is dimensionless – it is a scaling, not a unit.

The definition of Gain in decibel is:

decibels (dB) = $20 \log_{10} |V_2 / V_1|$

When expressing power gain, the formula is different – the constant before the log is 10x not 20x.



I now want to show you how to make informed guess to the magnitude response of a circuit from its gain equation without having to do any calculations. This gain equation is frequency dependent and is often written as H(jω). Since it is the ratio of output voltage to input voltage (or output current to input current), it is called a **TRANSFER FUNCTION**:

$$Y(j\omega) = H(j\omega)\, X(j\omega)$$

Now consider the function: $H(j\omega) = a \times j\omega + b$.

If the magnitude $|a\omega|$ is much small than $|b|$, $|H(j\omega)| \rightarrow |b|$.

If the magnitude $|a\omega|$ is much larger than $|b|$, $|H(j\omega)| \rightarrow |a\omega|$.

Let us take the RC circuit as shown here. For low frequencies, the magnitude is 1 or 0 dB.

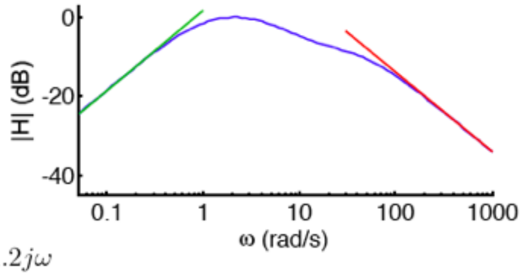
For high frequencies the magnitude drops linearly with ω (i.e. it is proportional to ω^{-1}. So, in terms of dBs, over one decade (factor of 10), it falls by 20dB. The slope of the line at high frequency is therefore -20dB/decade.

How high must the frequency be before it is called "high"? The intersection of these two line is at $\omega = 1/RC$. This is known as the corner frequency.

Low and High Frequency Asymptotes

- You can find the low and high frequency asymptotes without factorizing:

$$H(j\omega) = \frac{60(j\omega)^2+720(j\omega)}{3(j\omega)^3+165(j\omega)^2+762(j\omega)+600} = \frac{20j\omega(j\omega+12)}{(j\omega+1)(j\omega+4)(j\omega+50)}$$



Low Frequency Asymptote:

- From factors: $H_{LF}(j\omega) = \frac{20j\omega(12)}{(1)(4)(50)} = 1.2j\omega$
- Lowest power of jω on top and bottom: $H\,(j\omega) \simeq \frac{720(j\omega)}{600} = 1.2j\omega$

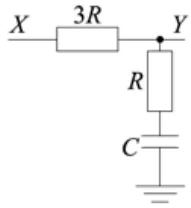
High Frequency Asymptote:

- From factors: $H_{HF}(j\omega) = \frac{20j\omega(j\omega)}{(j\omega)(j\omega)(j\omega)} = 20\,(j\omega)^{-1}$
- Highest power of jω on top and bottom: $H\,(j\omega) \simeq \frac{60(j\omega)^2}{3(j\omega)^3} = 20\,(j\omega)^{-1}$

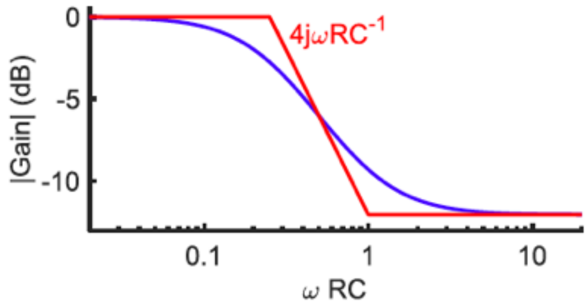
Here is another example of using the estimation method to plot the magnitude of the transfer function |H(jω)|. It does not matter where the function comes from, and it is complicated. Yet with our approximation method, you can plot the rough shape of the magnitude response.

RCR Circuit

$\frac{Y}{X} = \frac{R+\frac{1}{j\omega C}}{3R+R+\frac{1}{j\omega C}} = \frac{j\omega RC+1}{4j\omega RC+1}$



Corner freqs: $\frac{1}{4RC}^{-}, \frac{1}{RC}^{+}$ LF Asymptote: $H(j\omega)=1$



Magnitude Response:

Gradient Changes: -20 dB/dec at $\omega = \frac{1}{4RC}$ and $+20$ at $\omega = \frac{1}{RC}$

Line equations: $H(j\omega) =$ (a) 1, (b) $\frac{1}{4j\omega RC}$, (c) $\frac{j\omega RC}{4j\omega RC} = \frac{1}{4}$

Here is another example where we add a resistor R in series with C as shown.

Impedance of R and C in series is: $Z_{RC} = R + \frac{1}{j\omega C}$

Using voltage divider equation:

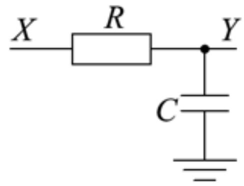
$$\frac{Y}{X} = \frac{Z_{RC}}{3R + Z_{RC}} = \frac{R + \frac{1}{j\omega C}}{4R + \frac{1}{j\omega C}}$$

Multiply top and bottom by jωC:

$$\frac{Y}{X} = \frac{1 + j\omega RC}{1 + j\omega 4RC}$$

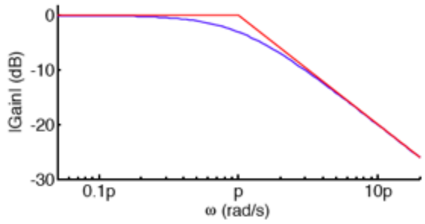
1st Order Low Pass Filter

$$\frac{Y}{X} = \frac{1/j\omega C}{R + 1/j\omega C} = \frac{1}{j\omega RC + 1} = \frac{1}{\frac{j\omega}{p} + 1}$$



- Corner frequency: $p = \left|\frac{b}{a}\right| = \frac{1}{RC}$
- Asymptotes: 1 and $\frac{p}{j\omega}$

  Very low ω: Capacitor = open circuit
  Very high ω: Capacitor = short circuit



- A low-pass filter because it allows low frequencies to pass but ***attenuates*** (makes smaller) high frequencies.
- The ***order*** of a filter: highest power of $j\omega$ in the denominator.
- Almost always equals the total number of L and/or C.

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Let us take another look at the simple RC network. Remember that the impedance of a capacitor is inversely proportional to frequency. Therefore at low frequency, a capacitor appears as open-circuit. At high frequency, it appears as short-circuit.

Using the principle of voltage divider, this circuit will give you a low output at high frequency (X_C is small), and does not attenuate the signal at low frequency (X_C is large). We call this a low pass filter (LF).

The transfer function H(jω) = Y/X has only one jω term in the denominator. We call this first order filter.

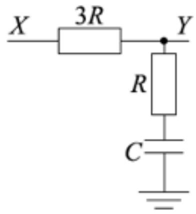
The order of a filter is the highest power of the (jω) term in the denominator of the transfer function (Gain function).

A first order filter will have a roll-off (i.e. rate of drop in gain) of -20dB/decade.

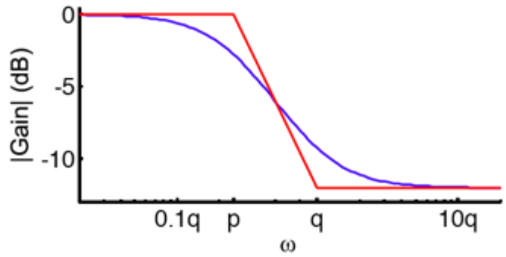
An nth order filter will have a roll-off of -20n dB/decade.

Low Pass Filter with Gain Floor

$$\frac{Y}{X} = \frac{R + 1/j\omega C}{4R + 1/j\omega C} = \frac{j\omega RC + 1}{j\omega 4RC + 1} = \frac{\frac{j\omega}{q} + 1}{\frac{j\omega}{p} + 1}$$



- Corner frequency: $p = \frac{1}{4RC}$, $q = \frac{1}{RC}$
- Asymptotes: 1 and ¼



Very low ω:

Capacitor = open circuit

Resistor R unattached. Gain = 1

Very high ω:

Capacitor short circuit

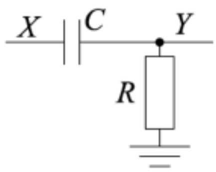
Circuit is potential divider with gain $20\log_{10}\frac{1}{4} = -12\,\text{dB}$.

We have seen this before. Here we consider it as a filter. If R = 0, it is similar to the simple RC network we have seen in the last slide. However, we have added a R in series with C.

Introducing the resistor R in series with C adds an extra corner frequency at q. Now the gain drops off initially at -20dB/decade after the first corner frequency at 1/4RC. However, at 1/RC, the gain levels out as the impedance of C is now negligible, and the circuit appears like a simple resistor voltage divider.

High Pass Filter

$$\frac{Y}{X} = \frac{R}{R + 1/j\omega C} = \frac{j\omega RC}{j\omega RC + 1}$$



- Corner frequency: $p = \frac{1}{RC}$
- Asymptotes: $jwRC$ and 1

Very low ω:

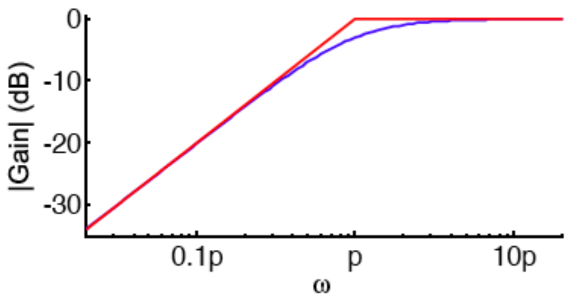
Capacitor = open circuit

Gain = 0

Very high ω:

Capacitor short circuit

Gain = 1



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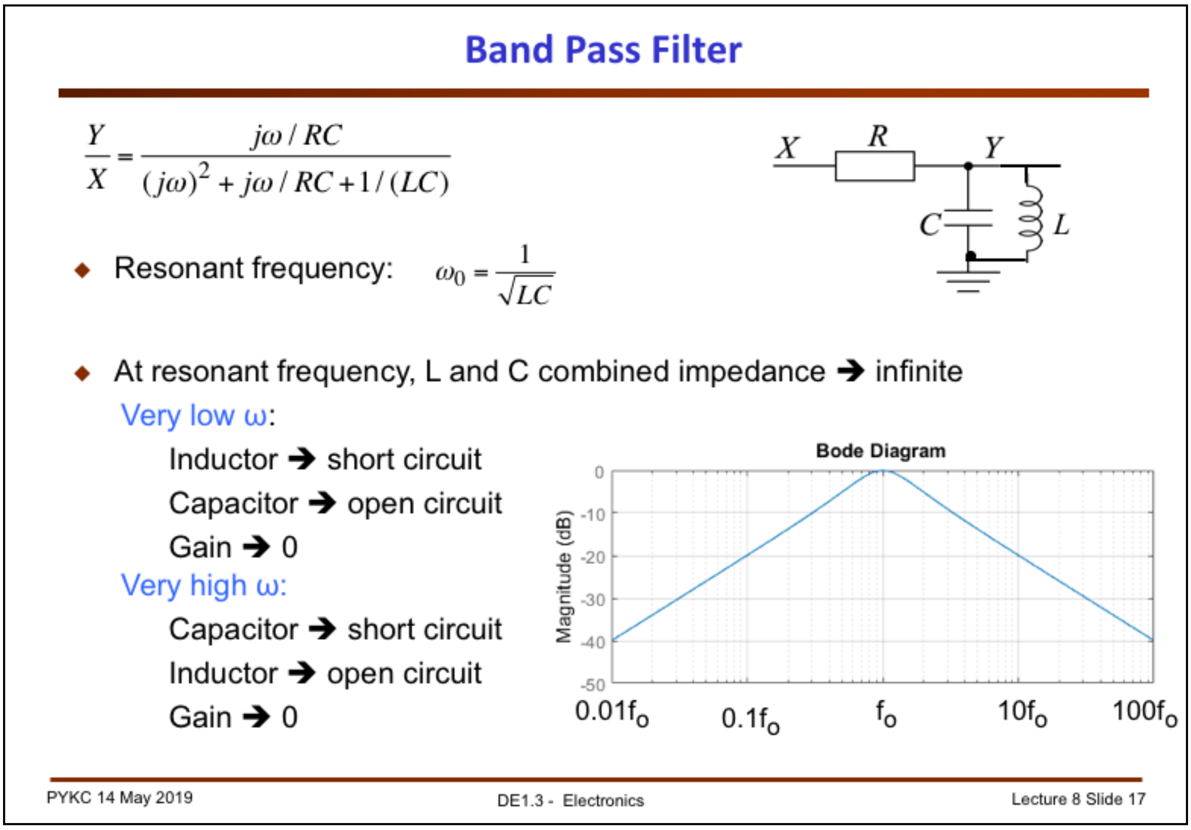
If you swap the RC and to form a CR circuit as shown, we have a magnitude response where at high frequency, C appears to be short circuit and Y=X. However C blocks any low frequency and DC signals. Therefore we now have a high pass filter.

Again you can work out the straight line approximation with the gain equation (transfer function).



This is a circuit including a R in series with L and C in parallel. I want you to try to derive the Y/X equation at home and prove that it is as given here.

I will not go into details for this circuit.

Generally speaking, the order of the filter is the number of C + number of L. So this circuit is a 2nd order filter. This is seen from the gain equation which has $(j\omega)^2$ term in the denominator.

The frequency at which the gain is maximum is called the resonant frequency. A this frequency, the impedance (and reactance) of L || C circuit is approaching infinity (open circuit). Therefore this network will pass through only signals at the resonant frequency and suppress signals of all other frequencies. It is called a band pass filter.